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New traveling wave solutions for higher dimensional nonlinear evolution equations using a generalized $(\frac{G'}{G})$-expansion method

E M E Zayed[1]

Mathematics Department, Faculty of Science, Zagazig University, Zagazig, Egypt

E-mail: emezayed@hotmail.com

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Abstract
In this paper, we construct new traveling wave solutions of some nonlinear evolution equations in mathematical physics via the (3+1)-dimensional potential-YTSF equation, the (3+1)-dimensional modified KdV–Zakharov–Kuznetsev equation, the (3+1)-dimensional Kadomtsev–Petviashvili equation and the (1+1)-dimensional KdV equation by using a generalized $\left(\frac{G'}{G}\right)$-expansion method, where $G = G(\xi)$ satisfies the Jacobi elliptic equation $[G'(\xi)]^2 = P(G)$. Here, we assume that $P(G)$ is a polynomial of fourth order. Many new exact solutions in terms of the Jacobi elliptic functions are obtained.

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1. Introduction

In recent years, the exact solutions of nonlinear PDEs have been investigated by many authors (see, for example, [1–54]) who are interested in nonlinear physical phenomena. Many powerful different methods have been presented by those authors. For integrable nonlinear differential equations, the inverse scattering transform method [3], the Hirota method [10], the truncated Painlevé expansion method [31, 46], the Backlund transform method [18, 19] and the exp-function method [6, 9, 36, 48, 49] are used in looking for exact solutions. Among non-integrable nonlinear differential equations there is a wide class of equations that are referred to as partially integrable equations because they become integrable for some values of their parameters. There are many different methods to look for the exact solutions of these equations. The most famous algorithms are the truncated Painlevé expansion method [14], the Weierstrass

[1] Present address: Mathematics Department, Faculty of Science, Taif University, El-Taif, El-Hawiyah, PO box 888, Kingdom of Saudi Arabia.

elliptic function method [13], the tanh-function method [1, 8, 41, 50] and the Jacobi elliptic function expansion method [7, 15, 17, 29, 32, 34, 37, 40, 42]. There are other methods which can be found in [2, 12, 16, 20–27, 33, 39, 43, 47].

Recently, Wang *et al* [28] have introduced a simple method, called the $\left(\frac{G'}{G}\right)$-expansion method, to look for traveling wave solutions of nonlinear evolution equations, where $G = G(\xi)$ satisfies the second-order linear ordinary differential equation $G''(\xi) + \lambda G'(\xi) + \mu G(\xi) = 0$; here λ and μ are arbitrary constants. For further references, see [5, 44, 45, 53, 54].

In the present paper, we shall use an alternative approach, which may be called a generalized $\left(\frac{G'}{G}\right)$-expansion method. The main idea of this method is that the traveling wave solutions of nonlinear differential equations can be expressed by a polynomial in $\left(\frac{G'}{G}\right)$, where $G = G(\xi)$ satisfies the Jacobi elliptic equation $[G'(\xi)]^2 = e_2G^4(\xi) + e_1G^2(\xi) + e_0$, instead of satisfying the second-order linear ordinary differential equation, where $\xi = x + y + z - Vt$ and e_2, e_1, e_0, V are arbitrary constants while $' = \frac{d}{d\xi}$. The degree of this polynomial can be determined by considering the homogeneous balance between the highest order derivatives and the nonlinear terms appearing in the given nonlinear equations. The coefficients of this polynomial can be obtained by solving a set of algebraic equations, which result from the process of using the proposed method. This approach will play an important role in constructing many new traveling wave solutions for the nonlinear PDEs via the (3+1)-dimensional potential-YTSF equation, the (3+1)-dimensional modified KdV–Zakharov–Kuznetsev equation, the (3+1)-dimensional Kadomtsev–Petviashvili equation and the (1+1)-dimensional KdV equation, in terms of the Jacobi elliptic functions.

2. Description of a generalized $\left(\frac{G'}{G}\right)$-expansion method

Suppose that we have the following nonlinear partial differential equation,

$$F(u, u_t, u_x, u_y, u_z, u_{tt}, u_{xt}, u_{xx}, u_{xy}, u_{yy}, u_{yt}, u_{zz}, u_{zt}, u_{zx}, u_{zy}, \ldots) = 0, \tag{2.1}$$

where $u = u(x, y, z, t)$ is an unknown function, F is a polynomial in $u(x, y, z, t)$ and its partial derivatives in which the highest order derivatives and the nonlinear terms are involved. In the following, we give the main steps of the generalized $\left(\frac{G'}{G}\right)$-expansion method.

Step 1. The traveling wave variable

$$u(x, y, z, t) = u(\xi), \qquad \xi = x + y + z - Vt, \tag{2.2}$$

where V is a constant, allows us to reduce equation (2.1) to an ODE for $u = u(\xi)$ in the form

$$P(u, u', u'', u''', \ldots) = 0. \tag{2.3}$$

Step 2. Suppose the solution of equation (2.3) can be expressed by a polynomial in $\left(\frac{G'}{G}\right)$ as follows,

$$u(\xi) = \sum_{i=0}^{n} \alpha_i \left(\frac{G'}{G}\right)^i, \tag{2.4}$$

where $G = G(\xi)$ satisfies the following Jacobi elliptic equation,

$$[G'(\xi)]^2 = e_2G^4(\xi) + e_1G^2(\xi) + e_0, \tag{2.5}$$

where α_i, e_2, e_1, e_0 and V are the arbitrary constants to be determined, provided $\alpha_n \neq 0$. The positive integer n can be determined by considering the homogeneous balance between the highest order derivatives and the nonlinear terms appearing in equation (2.1) or (2.3). More precisely, we define the degree of $u(\xi)$ as $D[u(\xi)] = n$, which gives rise to the degree of other expressions as follows,

$$D\left[\frac{\mathrm{d}^q u}{\mathrm{d}\xi^q}\right] = n + q, \qquad D\left[u^p\left(\frac{\mathrm{d}^q u}{\mathrm{d}\xi^q}\right)^s\right] = np + s(q + n). \tag{2.6}$$

Therefore, we can get the value of n in (2.4).

Step 3. Substituting (2.4) into (2.3) and using equation (2.5), we obtain polynomials in $G^j(\xi)$, $G'(\xi)G^j(\xi)$ $(j = \pm 1, \pm 2, \ldots)$. Equating each coefficient of the resulted polynomials to zero yields a set of algebraic equations for α_i, e_2, e_1, e_0 and V.

Step 4. Since the general solutions of equation (2.5) are well known to us (see appendix A), substituting α_i, V and the general solution of equation (2.5) into (2.4) we have many new traveling wave solutions of the nonlinear partial differential equation (2.1).

3. Some applications

In this section, we apply the generalized $\left(\frac{G'}{G}\right)$-expansion method to construct new traveling wave solutions for the (3+1)-dimensional potential-YTSF equation, the (3+1)-dimensional modified KdV–Zakharov–Kuznetsev equation, the (3+1)-dimensional Kadomtsev–Petviashvili equation and the (1+1)-dimensional KdV equation, which are very important nonlinear evolution equations in mathematical physics and have attracted the attention of many researchers.

3.1. Example 1. The (3+1)-dimensional potential-YTSF equation

We start with the (3+1)-dimensional potential-YTSF equation [30, 35, 38] of the form

$$-4u_{xt} + u_{xxxz} + 4u_x u_{xz} + 2u_{xx}u_z + 3u_{yy} = 0. \tag{3.1}$$

Yu *et al* [38] extended the Bogoyavlenskii–Schiff equation [30, 35, 38]

$$v_t + \phi(v)v_z = 0, \qquad \phi(v) = \partial_x^2 + 4v + 2v_x\partial_x^{-1}, \tag{3.2}$$

to be the new (3+1)-dimensional nonlinear evolution equation

$$(-4v_t + \phi(v)v_z)_x + 3v_{yy} = 0, \qquad \phi(v) = \partial_x^2 + 4v + 2v_x\partial_x^{-1}, \tag{3.3}$$

where $\partial_x^{-1} f = \int f\,\mathrm{d}x$, which is called the (3+1)-dimensional YTSF equation. Using the potential $v = u_x$ gives the (3+1)-dimensional potential-YTSF equation (3.1). The authors gave a traveling solitary wave solution of (3.3). The Backlund transformation and some soliton-like solutions for the potential form of (3.2) have also been found. Yan [35] has found an auto-Backlund transformation of equation (3.1) and has arrived at some families of exact soliton-like solutions and rational solutions as well.

Let us now solve equation (3.1) by the generalized $\left(\frac{G'}{G}\right)$-expansion method. To this end, we see that the following traveling wave variables,

$$u(x, y, z, t) = u(\xi), \tag{3.4}$$

where

$$\xi = x + y + z - Vt, \tag{3.5}$$

and V is a constant, permit us converting equation (3.1) into the following ODE

$$C + (3 + 4V)u' + u''' + 3u'^2 = 0, \tag{3.6}$$

and C is an integration constant. Suppose that the solution of equation (3.6) can be expressed by a polynomial in $\left(\frac{G'}{G}\right)$ as follows:

$$u(\xi) = \sum_{i=0}^{n} \alpha_i \left(\frac{G'}{G}\right)^i, \tag{3.7}$$

where α_i are arbitrary constants, while $G(\xi)$ satisfies the Jacobi elliptic equation (2.5).

Considering the homogeneous balance between the highest order derivative and the nonlinear term in (3.6), we deduce from (2.6) that $D(u''') = D(u'^2)$. Therefore $n+3 = 2(n+1)$ and hence $n = 1$. Thus, we obtain

$$u(\xi) = \alpha_1 \left(\frac{G'}{G}\right) + \alpha_0. \tag{3.8}$$

From (2.5) and (3.8) we deduce that

$$u' = \alpha_1 \left[\frac{G''}{G} - \left(\frac{G'}{G}\right)^2\right], \tag{3.9}$$

where

$$\frac{G''}{G} = e_1 + 2e_2 G^2. \tag{3.10}$$

Consequently, we have the following derivatives

$$u' = \alpha_1[e_2 G^2 - e_0 G^{-2}], \tag{3.11}$$

$$u'' = 2\alpha_1 G'[e_2 G + e_0 G^{-3}], \tag{3.12}$$

$$u''' = 2\alpha_1 \left[2e_1 e_2 G^2 - 2e_1 e_0 G^{-2} + 3e_2^2 G^4 - 3e_0^2 G^{-4}\right], \tag{3.13}$$

and so on.

Substituting (3.11) and (3.13) into (3.6) we get the following polynomial

$$G^2[4\alpha_1 e_1 e_2 + (4V+3)\alpha_1 e_2] + G^{-2}[-(4V+3)\alpha_1 e_0 - 4\alpha_1 e_1 e_0] + G^4\left[6\alpha_1 e_2^2 + 3\alpha_1^2 e_2^2\right] + G^{-4}\left[-6\alpha_1 e_0^2 + 3\alpha_1^2 e_0^2\right] + C - 6\alpha_1^2 e_2 e_0 = 0. \tag{3.14}$$

Consequently, we have the following system of algebraic equations

$$\begin{aligned}
&4\alpha_1 e_1 e_2 + (4V+3)\alpha_1 e_2 = 0,\\
&-(4V+3)\alpha_1 e_0 - 4\alpha_1 e_1 e_0 = 0,\\
&6\alpha_1 e_2^2 + 3\alpha_1^2 e_2^2 = 0,\\
&-6\alpha_1 e_0^2 + 3\alpha_1^2 e_0^2 = 0,\\
&C - 6\alpha_1^2 e_2 e_0 = 0,
\end{aligned} \tag{3.15}$$

which can be solved to obtain

$$\alpha_1 = -2, \qquad V = -\left(e_1 + \tfrac{3}{4}\right), \qquad e_0 = 0, \qquad C = 0. \tag{3.16}$$

Substituting (3.16) into (3.8) yields

$$u(\xi) = -2\left(\frac{G'}{G}\right) + \alpha_0, \tag{3.17}$$

where

$$\xi = x + y + z + t\left(e_1 + \tfrac{3}{4}\right). \tag{3.18}$$

According to appendix A, we have the following families of exact solutions.

Family 1. If $e_0 = 0, e_1 = 1, e_2 = -1$, then we obtain

$$u(\xi) = 2\tanh(\xi) + \alpha_0, \tag{3.19}$$

where

$$\xi = x + y + z + \tfrac{7}{4}t. \tag{3.20}$$

Family 2. If $e_0 = 0, e_1 = e_2 = 1$, then we obtain

$$u(\xi) = 2\coth(\xi) + \alpha_0, \tag{3.21}$$

where

$$\xi = x + y + z + \tfrac{7}{4}t. \tag{3.22}$$

Family 3. If $e_0 = 0, e_1 = -1, e_2 = 1$, then we obtain

$$u(\xi) = -2\tan(\xi) + \alpha_0, \tag{3.23}$$

where

$$\xi = x + y + z - \tfrac{1}{4}t. \tag{3.24}$$

Family 4. If $e_0 = e_1 = 0, e_2 = 1$, then we obtain

$$u(\xi) = \frac{2}{\xi} + \alpha_0, \tag{3.25}$$

where

$$\xi = x + y + z + \tfrac{3}{4}t. \tag{3.26}$$

Family 5. If $e_0 = 0, e_1 = -(m^2 + 1), e_2 = m^2$, then we obtain

$$u(\xi) = -2cs(\xi)dn(\xi) + \alpha_0, \tag{3.27}$$

where

$$\xi = x + y + z - t\left(m^2 + \tfrac{1}{4}\right). \tag{3.28}$$

3.2. Example 2. The (3+1)-dimensional modified KdV–Zakharov–Kuznetsev equation

In this subsection, we consider the (3+1)-dimensional modified KdV–Zakharov–Kuznetsev equation [32] in the form

$$u_t + \alpha u^2 u_x + u_{xxx} + u_{xyy} + u_{xzz} = 0, \tag{3.29}$$

where α is a nonzero constant. Xu [32] has discussed equation (3.29) using an elliptic equation method and found new types of elliptic function solutions.

Let us now solve equation (3.29) by the proposed method. To this end, we see that the traveling wave variable (3.4) allows us to convert equation (3.29) into the following ODE,

$$C - Vu + \tfrac{1}{3}\alpha u^3 + 3u'' = 0, \tag{3.30}$$

where C is a constant of integration. Considering the homogeneous balance between the highest order derivative and the nonlinear term in (3.30), we get $n = 1$. Thus, we have

$$u(\xi) = \alpha_1\left(\frac{G'}{G}\right) + \alpha_0. \tag{3.31}$$

Substituting (3.9) and (3.31) into (3.30) along with (2.5), we get the following polynomial:

$$G^2\big[\alpha\alpha_1^2\alpha_0 e_2\big] + G^{-2}\big[\alpha\alpha_1^2\alpha_0 e_0\big] + GG'\left[\frac{\alpha}{3}\alpha_1^3 e_2 + 6\alpha_1 e_2\right] + G^{-1}G'\left[-V\alpha_1 + \frac{\alpha}{3}\alpha_1^3 e_1 + \alpha\alpha_1\alpha_0^2\right] + G^{-3}G'\left[\frac{\alpha}{3}\alpha_1^3 e_0 + 6\alpha_1 e_0\right] + C - V\alpha_0 + \frac{\alpha}{3}\alpha_0^3 + \alpha\alpha_1^2 e_1\alpha_0 = 0. \qquad (3.32)$$

Consequently, we have the following system of algebraic equations,

$$\alpha\alpha_1^2\alpha_0 e_2 = 0,$$
$$\alpha\alpha_1^2\alpha_0 e_0 = 0,$$
$$\frac{\alpha}{3}\alpha_1^3 e_2 + 6\alpha_1 e_2 = 0,$$
$$-V\alpha_1 + \frac{\alpha}{3}\alpha_1^3 e_1 + \alpha\alpha_1\alpha_0^2 = 0,$$
$$\frac{\alpha}{3}\alpha_1^3 e_0 + 6\alpha_1 e_0 = 0,$$
$$C - V\alpha_0 + \frac{\alpha}{3}\alpha_0^3 + \alpha\alpha_1^2 e_1\alpha_0 = 0,$$

which can be solved to get

$$\alpha_1 = \pm 3\sqrt{\frac{-2}{\alpha}}, \qquad \alpha_0 = 0, \qquad V = -6e_1, \qquad C = 0. \qquad (3.33)$$

Substituting (3.33) into (3.31) yields

$$u(\xi) = \pm 3\sqrt{\frac{-2}{\alpha}}\left(\frac{G'}{G}\right), \qquad (3.34)$$

where

$$\xi = x + y + z + 6e_1 t.$$

According to appendix A, we write down only the first three families of exact solutions for equation (3.29) as follows.

Family 1. If $e_0 = 1$, $e_1 = -(m^2+1)$, $e_2 = m^2$, then we get

$$u(\xi) = \pm 3\sqrt{\frac{-2}{\alpha}}cs(\xi)dn(\xi),$$

or

$$u(\xi) = \mp 3\sqrt{\frac{-2}{\alpha}}(1-m^2)sd(\xi)nc(\xi), \qquad (3.35)$$

where $\xi = x + y + z - 6t(m^2+1)$.

Family 2. If $e_0 = 1 - m^2$, $e_1 = 2m^2 - 1$, $e_2 = -m^2$, then we get

$$u(\xi) = \mp 3\sqrt{\frac{-2}{\alpha}}sc(\xi)dn(\xi), \qquad (3.36)$$

where $\xi = x + y + z + 6t(2m^2 - 1)$.

Family 3. If $e_0 = m^2 - 1$, $e_1 = 2 - m^2$, $e_2 = -1$, then we get

$$u(\xi) = \mp 3\sqrt{\frac{-2}{\alpha}}m^2 sn(\xi)cd(\xi), \qquad (3.37)$$

where $\xi = x + y + z + 6t(2 - m^2)$.

Similarly, we can write down the other families of exact solutions of equation (3.29) which are omitted for convenience.

3.3. Example 3. The (3+1)-dimensional Kadomtsev–Petviashvili equation

In this subsection, we consider the (3+1)-dimensional Kadomtsev–Petviashvili equation [11, 51, 52] in the form

$$u_{xt} + 6(u_x)^2 + 6uu_{xx} - u_{xxxx} - u_{yy} - u_{zz} = 0, \quad (3.38)$$

which describes the dynamics of solitons and nonlinear waves in plasmas and superfluids. Recently, Zhang [48] used the exp-function method to obtain generalized solitonary solutions and periodic solutions of equation (3.38). Solitary wave solutions, Jacobi elliptic functions solutions, soliton-like solutions and other types of exact solutions of equation (3.38) can be found in [11, 51, 52].

Let us now solve this equation by the proposed method. To this end, we see that the traveling wave variable (3.4) allows us to convert (3.38) into the following ODE,

$$-(V+2)u + 3u^2 - u'' = 0, \quad (3.39)$$

where the constants of integration are assumed to be zero. Considering the homogeneous balance between the highest order derivative and the nonlinear term in (3.39), we get $n = 2$. Thus, we have

$$u(\xi) = \alpha_2\left(\frac{G'}{G}\right)^2 + \alpha_1\left(\frac{G'}{G}\right) + \alpha_0. \quad (3.40)$$

Consequently, we deduce from (2.5) and (3.40) that

$$u' = 2\alpha_2 G'[e_2 G - e_0 G^{-3}] + \alpha_1[e_2 G^2 - e_0 G^{-2}], \quad (3.41)$$

$$u'' = 2\alpha_2\left[2e_1e_2G^2 + 2e_1e_0G^{-2} + 3e_2^2G^4 + 3e_0^2G^{-4} + 2e_0e_2\right] + 2\alpha_1 G'[e_2G + e_0G^{-3}]. \quad (3.42)$$

Substituting (3.40) and (3.42) into (3.39) we get the following polynomial:

$$\begin{aligned}
&G^2\left[-\alpha_2e_2(V+2) + 6\alpha_2^2e_1e_2 + 3\alpha_1^2e_2 + 6\alpha_2\alpha_0e_2 - 4\alpha_2e_1e_2\right]\\
&\quad + G^{-2}\left[-\alpha_2e_0(V+2) + 6\alpha_2^2e_1e_0 + 3\alpha_1^2e_0 + 6\alpha_2e_0\alpha_0 - 4\alpha_2e_1e_0\right]\\
&\quad + G^4\left[3\alpha_2^2e_2^2 - 6\alpha_2e_2^2\right] + G^{-4}\left[3\alpha_2^2e_0^2 - 6\alpha_2e_0^2\right]\\
&\quad + GG'[6\alpha_2\alpha_1e_2 - 2\alpha_1e_2] + G^{-1}G'[-\alpha_1(V+2) + 6\alpha_2\alpha_1e_1 + 6\alpha_1\alpha_0]\\
&\quad + G^{-3}G'[6\alpha_2\alpha_1e_0 - 2\alpha_1e_0] + 6\alpha_2e_1\alpha_0 - (V+2)(\alpha_2e_1 + \alpha_0) + 3\alpha_2^2e_1^2\\
&\quad + 6\alpha_2^2e_2e_0 + 3\alpha_1^2e_1 + 3\alpha_0^2 - 4\alpha_2e_2e_0 = 0.
\end{aligned} \quad (3.43)$$

Consequently, we have the following system of algebraic equations,

$$\begin{aligned}
&-\alpha_2e_2(V+2) + 6\alpha_2^2e_1e_2 + 3\alpha_1^2e_2 + 6\alpha_2\alpha_0e_2 - 4\alpha_2e_1e_2 = 0,\\
&-\alpha_2e_0(V+2) + 6\alpha_2^2e_1e_0 + 3\alpha_1^2e_0 + 6\alpha_2e_0\alpha_0 - 4\alpha_2e_1e_0 = 0,\\
&3\alpha_2^2e_2^2 - 6\alpha_2e_2^2 = 0,\\
&3\alpha_2^2e_0^2 - 6\alpha_2e_0^2 = 0,\\
&6\alpha_2\alpha_1e_2 - 2\alpha_1e_2 = 0,\\
&-\alpha_1(V+2) + 6\alpha_2\alpha_1e_1 + 6\alpha_1\alpha_0 = 0,\\
&6\alpha_2\alpha_1e_0 - 2\alpha_1e_0 = 0,\\
&6\alpha_2\alpha_0e_1 - (V+2)(\alpha_2e_1 + \alpha_0) + 3\alpha_2^2e_1^2 + 6\alpha_2^2e_2e_0 + 3\alpha_1^2e_1 + 3\alpha_0^2 - 4\alpha_2e_2e_0 = 0,
\end{aligned}$$

which can be solved to get

$$\alpha_2 = 2, \qquad \alpha_1 = 0, \qquad V = 8e_1 + 6\alpha_0 - 2, \quad (3.44)$$

where

$$\alpha_0 = -\frac{4}{3}e_1 \mp \frac{2}{3}\sqrt{e_1^2 - 12e_2e_0}.$$

Substituting (3.44) into (3.40) yields

$$u(\xi) = 2\left(\frac{G'}{G}\right)^2 - \frac{4}{3}e_1 \mp \frac{2}{3}\sqrt{e_1^2 - 12e_2e_0}, \tag{3.45}$$

where

$$\xi = x + y + z - t(8e_1 + 6\alpha_0 - 2).$$

According to appendix A, we write down only the first three families of exact solutions for equation (3.38) as follows.

Family 1. If $e_0 = 1, e_1 = -(m^2 + 1), e_2 = m^2$, then we get

$$u(\xi) = 2cn^2(\xi)ds^2(\xi) + \tfrac{4}{3}(m^2 + 1) \mp \tfrac{2}{3}\sqrt{m^4 - 10m^2 + 1}, \tag{3.46}$$

or

$$u(\xi) = 2(1 - m^2)^2 sd^2(\xi)nc^2(\xi) + \tfrac{4}{3}(m^2 + 1) \mp \tfrac{2}{3}\sqrt{m^4 - 10m^2 + 1}, \tag{3.47}$$

where $\xi = x + y + z + 2t[1 \pm 2\sqrt{m^4 - 10m^2 + 1}]$.

Family 2. If $e_0 = 1 - m^2, e_1 = 2m^2 - 1, e_2 = -m^2$, then we get

$$u(\xi) = 2sc^2(\xi)dn^2(\xi) - \tfrac{4}{3}(2m^2 - 1) \mp \tfrac{2}{3}\sqrt{8m^2 - 8m^4 + 1}, \tag{3.48}$$

where $\xi = x + y + z + 2t[1 \pm 2\sqrt{8m^2 - 8m^4 + 1}]$.

Family 3. If $e_0 = m^2 - 1, e_1 = 2 - m^2, e_2 = -1$, then we get

$$u(\xi) = 2m^4 sn^2(\xi)cd^2(\xi) - \tfrac{4}{3}(2 - m^2) \mp \tfrac{2}{3}\sqrt{m^4 + 8m^2 - 8}, \tag{3.49}$$

where $\xi = x + y + z + 2t[1 \pm 2\sqrt{m^4 + 8m^2 - 8}]$. Similarly, we can write down the other families of exact solutions of equation (3.38) which are omitted for convenience.

3.4. Example 4. The (1+1)-dimensional KdV equation

In this subsection, we consider the following famous (1+1)-dimensional KdV equation:

$$u_t + 6uu_x + u_{xxx} = 0. \tag{3.50}$$

This equation is a model that governs the one-dimensional propagation of small amplitude, weakly dispersive waves, and plays a major role in the soliton concepts.

Let us now solve this equation by the proposed method. To this end, we see that the traveling wave variable $u(x,t) = u(\xi), \xi = x - Vt$, allows us to convert (3.50) into the following ODE,

$$-Vu + 3u^2 + u'' = 0, \tag{3.51}$$

where the constants of integration are assumed to be zero. Considering the homogeneous balance between the highest order derivative and the nonlinear term in (3.51), we get $n = 2$. Thus, the solution of equation (3.50) has the same form as (3.40). Substituting (3.40) and

(3.42) into (3.51) we get the following polynomial:

$$\begin{aligned}
&G^2\left[-\alpha_2 e_2 V + 6\alpha_2^2 e_1 e_2 + 3\alpha_1^2 e_2 + 6\alpha_2\alpha_0 e_2 + 4\alpha_2 e_1 e_2\right] \\
&\quad + G^{-2}\left[-\alpha_2 e_0 V + 6\alpha_2^2 e_1 e_0 + 3\alpha_1^2 e_0 + 6\alpha_2 e_0\alpha_0 + 4\alpha_2 e_1 e_0\right] \\
&\quad + G^4\left[3\alpha_2^2 e_2^2 + 6\alpha_2 e_2^2\right] + G^{-4}\left[3\alpha_2^2 e_0^2 + 6\alpha_2 e_0^2\right] \\
&\quad + GG'[6\alpha_2\alpha_1 e_2 + 2\alpha_1 e_2] + G^{-1}G'[-\alpha_1 V + 6\alpha_2\alpha_1 e_1 + 6\alpha_1\alpha_0] \\
&\quad + G^{-3}G'[6\alpha_2\alpha_1 e_0 + 2\alpha_1 e_0] + 6\alpha_2 e_1\alpha_0 - V(\alpha_2 e_1 + \alpha_0) + 3\alpha_2^2 e_1^2 \\
&\quad + 6\alpha_2^2 e_2 e_0 + 3\alpha_1^2 e_1 + 3\alpha_0^2 + 4\alpha_2 e_2 e_0 = 0.
\end{aligned} \tag{3.52}$$

On equating the coefficients of (3.52) to zero and solving the resulting algebraic equations, we have

$$\alpha_2 = -2, \qquad \alpha_1 = 0, \qquad V = -8e_1 + 6\alpha_0, \tag{3.53}$$

where α_0 is given by the formula

$$\alpha_0 = \tfrac{4}{3}e_1 \mp \tfrac{2}{3}\sqrt{e_1^2 - 12e_2e_0}.$$

Substituting (3.53) into (3.40) yields

$$u(\xi) = -2\left(\frac{G'}{G}\right)^2 + \frac{4}{3}e_1 \mp \frac{2}{3}\sqrt{e_1^2 - 12e_2e_0}, \tag{3.54}$$

where

$$\xi = x + t(8e_1 - 6\alpha_0). \tag{3.55}$$

According to appendix A, we write down only the first three families of exact solutions for equation (3.50) as follows.

Family 1. If $e_0 = 1, e_1 = -(m^2+1), e_2 = m^2$, then we get

$$u(\xi) = -2cn^2(\xi)ds^2(\xi) - \tfrac{4}{3}(m^2+1) \mp \tfrac{2}{3}\sqrt{m^4 - 10m^2 + 1}, \tag{3.56}$$

or

$$u(\xi) = -2(1-m^2)^2 sd^2(\xi)nc^2(\xi) - \tfrac{4}{3}(m^2+1) \mp \tfrac{2}{3}\sqrt{m^4 - 10m^2 + 1}, \tag{3.57}$$

where $\xi = x \pm 4t\sqrt{m^4 - 10m^2 + 1}$.

Family 2. If $e_0 = 1 - m^2, e_1 = 2m^2 - 1, e_2 = -m^2$, then we get

$$u(\xi) = -2sc^2(\xi)dn^2(\xi) + \tfrac{4}{3}(2m^2 - 1) \mp \tfrac{2}{3}\sqrt{8m^2 - 8m^4 + 1}, \tag{3.58}$$

where $\xi = x \pm 4t\sqrt{8m^2 - 8m^4 + 1}$.

Family 3. If $e_0 = m^2 - 1, e_1 = 2 - m^2, e_2 = -1$, then we get

$$u(\xi) = -2m^4 sn^2(\xi)cd^2(\xi) + \tfrac{4}{3}(2 - m^2) \mp \tfrac{2}{3}\sqrt{m^4 + 8m^2 - 8}, \tag{3.59}$$

where $\xi = x \pm 4t\sqrt{m^4 + 8m^2 - 8}$.

Similarly, we can write down the other families of exact solutions of equation (3.50) which are omitted for convenience.

4. Conclusions

The main idea of the $\left(\frac{G'}{G}\right)$-expansion method (see [5, 28, 44, 45, 53, 54]) is that the traveling wave solutions of nonlinear partial differential equations can be expressed as polynomials in $\left(\frac{G'}{G}\right)$, where $G(\xi)$ satisfies a second-order linear ordinary differential equation. In the present paper, we have developed this method where we have assumed that $G(\xi)$ satisfies the Jacobi elliptic equation (2.5) instead of the standard technique used by Wang *et al* [28]. We have applied this alternative method to some nonlinear PDEs in mathematical physics via the (3+1)-dimensional potential-YTSF equation, the (3+1)-dimensional modified KdV–Zakharov–Kuznetsev equation, the (3+1)-dimensional Kadomtsev–Petviashvili equation and the (1+1)-dimensional KdV equation. We have obtained families of exact solutions of these equations in terms of the Jacobi elliptic functions.

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Appendix A

The general solutions to the Jacobi elliptic equation (2.5) and its derivatives (see, for example, [7, 15, 17, 34]) are listed as follows:

e_0	e_1	e_2	$G(\xi)$	$G'(\xi)$
1	$-(1+m^2)$	m^2	$sn(\xi)$ or $cd(\xi)$	$cn(\xi)dn(\xi)$; $-(1-m^2)sd(\xi)nd(\xi)$
$1-m^2$	$2m^2-1$	$-m^2$	$cn(\xi)$	$-sn(\xi)dn(\xi)$
m^2-1	$2-m^2$	-1	$dn(\xi)$	$-m^2sn(\xi)cn(\xi)$
m^2	$-(m^2+1)$	1	$ns(\xi)$ or $dc(\xi)$	$-ds(\xi)cs(\xi)$; $(1-m^2)nc(\xi)sc(\xi)$
$-m^2$	$2m^2-1$	$1-m^2$	$nc(\xi)$	$sc(\xi)dc(\xi)$
-1	$2-m^2$	m^2-1	$nd(\xi)$	$m^2sd(\xi)cd(\xi)$
$1-m^2$	$2-m^2$	1	$cs(\xi)$	$-ns(\xi)ds(\xi)$
1	$2-m^2$	$1-m^2$	$sc(\xi)$	$nc(\xi)dc(\xi)$
1	$2m^2-1$	$m^2(m^2-1)$	$sd(\xi)$	$nd(\xi)cd(\xi)$
$m^2(m^2-1)$	$2m^2-1$	1	$ds(\xi)$	$-cs(\xi)ns(\xi)$
$\frac{1}{4}$	$\frac{1}{2}(1-2m^2)$	$\frac{1}{4}$	$ns(\xi)\pm cs(\xi)$	$-ds(\xi)cs(\xi)\mp ns(\xi)ds(\xi)$
$\frac{1}{4}(1-m^2)$	$\frac{1}{2}(1+m^2)$	$\frac{1}{4}(1-m^2)$	$nc(\xi)\pm sc(\xi)$	$sc(\xi)dc(\xi)\pm nc(\xi)dc(\xi)$
$\frac{m^2}{4}$	$\frac{1}{2}(m^2-2)$	$\frac{1}{4}$	$ns(\xi)\pm ds(\xi)$	$-ds(\xi)cs(\xi)\mp cs(\xi)ns(\xi)$
$\frac{m^2}{4}$	$\frac{1}{2}(m^2-2)$	$\frac{m^2}{4}$	$sn(\xi)\pm \mathrm{i}cn(\xi)$	$cn(\xi)dn(\xi)\mp \mathrm{i}sn(\xi)dn(\xi)$
0	1	-1	$\mathrm{sech}(\xi)$	$-\mathrm{sech}(\xi)\tanh(\xi)$
0	1	1	$\mathrm{csch}(\xi)$	$-\mathrm{csch}(\xi)\coth(\xi)$
0	-1	1	$\sec(\xi)$	$\sec(\xi)\tan(\xi)$
0	0	1	$\frac{1}{\xi}$	$-\frac{1}{\xi^2}$
0	$-(1+m^2)$	m^2	$sn(\xi)$	$cn(\xi)dn(\xi)$

where $0 < m < 1$ is the modulus of the Jacobi elliptic functions and $\mathrm{i}=\sqrt{-1}$.

Appendix B

The Jacobi elliptic functions $sn(\xi)$, $cn(\xi)$, $dn(\xi)$, $ns(\xi)$, $cs(\xi)$, $ds(\xi)$, $sc(\xi)$, $sd(\xi)$ degenerate into hyperbolic functions when $m \to 1$ as follows,

$$sn(\xi) \longrightarrow \tanh(\xi), \quad cn(\xi) \longrightarrow \mathrm{sech}(\xi), \quad dn(\xi) \longrightarrow \mathrm{sech}(\xi), \quad ns(\xi) \longrightarrow \coth(\xi),$$
$$cs(\xi) \longrightarrow \mathrm{cosech}(\xi), \quad ds\xi) \longrightarrow \mathrm{cosech}(\xi), \quad sc(\xi) \longrightarrow \sinh(\xi), \quad sd(\xi) \longrightarrow \sinh(\xi),$$

and into trigonometric functions when $m \to 0$ as follows,

$$sn(\xi) \longrightarrow \sin(\xi), \quad cn(\xi) \longrightarrow \cos(\xi), \quad dn(\xi) \longrightarrow 1, \quad ns(\xi) \longrightarrow \mathrm{cosec}(\xi),$$
$$cs(\xi) \longrightarrow \cot(\xi), \quad ds(\xi) \longrightarrow \mathrm{cosec}(\xi), \quad sc(\xi) \longrightarrow \tan(\xi), \quad sd(\xi) \longrightarrow \sin(\xi).$$

Appendix C

$$cd(\xi) = \frac{cn(\xi)}{dn(\xi)}, \quad dc(\xi) = \frac{dn(\xi)}{cn(\xi)}, \quad nc(\xi) = \frac{1}{cn(\xi)}, \quad nd(\xi) = \frac{1}{dn(\xi)},$$
$$cs(\xi) = \frac{cn(\xi)}{sn(\xi)}, \quad sc(\xi) = \frac{sn(\xi)}{cn(\xi)}, \quad sd(\xi) = \frac{sn(\xi)}{dn(\xi)}, \quad ds(\xi) = \frac{dn(\xi)}{sn(\xi)}.$$

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